2.6.Fofonoff's analysis of interior transport:

In order to analyze the vertically integrated equations of motion start with the non accelerated equations of motion with vertical friction terms as considered by Sverdrup in equations (32):

$$-f\rho v = -\frac{\partial p}{\partial x} + \frac{\partial \tau_x}{\partial z}$$

$$f\rho u = -\frac{\partial p}{\partial y} + \frac{\partial \tau_y}{\partial z}$$

Expressing in terms of mass transports and On integrating these equations from ocean bottom (D) to the sea surface(η), we get:

$$-\int_D^\eta f\rho v\,dz = -\int_D^\eta \frac{\partial p}{\partial x}dz + \int_D^\eta \partial \tau_x$$

$$\int_D^\eta f\rho u\,dz = -\int_D^\eta \frac{\partial p}{\partial y}dz + \int_D^\eta \partial \tau_y \quad \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

The whole exercise of these equations to understand the transport in the subsurface layer due to the effect of wind stress at the surface lies on the evaluation of the first term on the right hand side. Fofonoff (1962) did extensive analysis. His analysis is as below:

Evaluation of $\int_D^\eta \frac{\partial p}{\partial x}dz$:

Put $E_p = \int_D^\eta p\,dz$ ……………………..(45)

where E_P is the potential energy of a column of water of unit horizontal area, relative the bottom of the ocean. Then we can write:

$$E_p = \int_D^\eta \frac{p\alpha}{g\alpha} g\,dz = \frac{1}{g}\int_D^\eta (p\alpha)\rho g\,dz = \frac{1}{g}\int_D^\eta (p\alpha)(-dp) = -\frac{1}{g}\int_D^\eta (p\alpha)(dp) = \frac{1}{g}\int_\eta^D (p\alpha)(dp)$$

$$E_P = \frac{1}{g}\int_\eta^D (p\alpha)(dp) = \frac{1}{g}\int_\eta^D (p\alpha_{35,0,p})(dp) + \frac{1}{g}\int_\eta^D (p\delta)(dp) = E_p^0 + \chi \quad \ldots\ldots\ldots\ldots(46)$$

Here χ is anomaly of potential energy and E_p^0 is the potential energy which is a function of bottom pressure P_D only.

$$\therefore \frac{\partial E_P}{\partial x} = \frac{\partial E_P^0}{\partial x} + \frac{\partial \chi}{\partial x} \quad \text{..................(47)}$$

From equation (45) we can write:

$$\frac{\partial}{\partial x}\left(E_p\right) = \frac{\partial}{\partial x}\int_D^\eta p dz = \int_D^\eta \frac{\partial p}{\partial x} dz + p_\eta \frac{\partial \eta}{\partial x} - p_D \frac{\partial D}{\partial x} \quad \text{.............(48)}$$

As η and D vary with x the last two terms in equation (48) appear. But it may be noted that P_η is pressure at sea surface is zero. So the second term in the right hand side of equation(48) vanishes. Hence equation (48)can be written as:

$$\int_D^\eta \frac{\partial p}{\partial x} dz = \frac{\partial}{\partial x}\left(E_p\right) + p_D \frac{\partial D}{\partial x} \quad \text{..............(49)}$$

Substituting (47) in (49) we can get

$$\int_D^\eta \frac{\partial p}{\partial x} dz = \frac{\partial E_P^0}{\partial x} + \frac{\partial \chi}{\partial x} + p_D \frac{\partial D}{\partial x} \quad \text{................(50)}$$

Fofonoff (1962) combined these two terms ($\frac{\partial E_P^0}{\partial x} + p_D \frac{\partial D}{\partial x}$) and expressed as

$$\int_D^\eta \frac{\partial p}{\partial x} dz = \frac{\partial \chi}{\partial x} + \frac{p_D \alpha_D}{g} \frac{\partial P_D}{\partial x} \quad \text{...................(51)}$$

Where $\frac{\partial P_D}{\partial x}$ is a component of pressure gradient along a level surface at the bottom and $\alpha_{35,0,p} P_D$ is assumed to be equal to the specific volume at the bottom, α_D.

Substituting equation (51) in equations (44) we get:

$$\begin{aligned} -\int_D^\eta f \rho v dz &= -\frac{\partial \chi}{\partial x} - \frac{p_D \alpha_D}{g} \frac{\partial P_D}{\partial x} + \tau_{x\eta} - \tau_{xD} \\ \int_D^\eta f \rho u dz &= -\frac{\partial \chi}{\partial y} - \frac{p_D \alpha_D}{g} \frac{\partial P_D}{\partial y} + \tau_{y\eta} - \tau_{yD} \end{aligned} \quad \text{............(52)}$$

i) **First terms on the right hand side of the equations (52):**

The first terms on the R.H.S are associated with variations in density and so these can be interpreted as the baroclinic part of the geostrophic velocity. So this is like balance between pressure gradient of mass transport to the coriolis part of the transport $\therefore -fM_{yc} = -\frac{\partial \chi}{\partial x}$ and $fM_{xc} = -\frac{\partial \chi}{\partial y}$ ....(53)

ii) **Second terms on the right hand side of the equations (52):**

The mass transports near the bottom are assumed as geostrophic as velocities near the bottom are small and tangential stresses near the bottom are neglected. So the pressure gradient at the bottom is equal to the coriolis at bottom.

$$\therefore fu_D = -\alpha_D \frac{\partial P_D}{\partial y} \quad \text{and} \quad fv_D = -\alpha_D \frac{\partial P_D}{\partial x} \quad \ldots.(54)$$

So the term $-\frac{p_D \alpha_D}{g}\frac{\partial P_D}{\partial x}$ can be written, substituting eqn (54) as

$$\frac{fp_D v_D}{g} \quad \ldots\ldots\ldots\ldots\ldots..(55)$$

This equation (55) can be related to fM_{yD} because $M_{yD} = \int_D^\eta \rho v_D dz = v_D \int_D^\eta \rho dz$ as v_D does not vary with depth. Taking ρdz = -dp/g, this becomes

$$v_D \int_D^\eta \rho dz = -v_D \int_D^\eta \frac{dp}{g} dz = -\frac{v_D}{g}\left(p_\eta - p_D\right) = \frac{v_D p_D}{g} (\because p_\eta = 0) \ldots\ldots..(56)$$

So from 55 & 56 we can write $fM_{yD} = \frac{fp_D v_D}{g}$ ....(57)

Similarly y component of the second term can be written as

$$fM_{xD} = \frac{fp_D u_D}{g} \ldots\ldots...(58)$$

iii) **Third terms on the right hand side of the equations (52):**

According to Ekman these terms can be written from equation (29a) as: $M_{xDE} = \frac{\tau_{yD}}{f}$ and $M_{yDE} = -\frac{\tau_{xD}}{f}$ ....(59)

So finally equations (52) for total transport can be written using equations (53), (57) and (59) as:

$$-fM_y = -fM_{yc} - fM_{yD} - fM_{yDE}$$

$$fM_x = fM_{xc} + fM_{xD} + fM_{xDE}$$

Which means

$$M_y = M_{yc} + M_{yD} + M_{yDE} \quad \ldots\ldots\ldots\ldots\ldots..(60)$$

$$M_x = M_{xc} + M_{xD} + M_{xDE}$$

Thus the total transport in the interior of the ocean can be considered to be made up of three types of transports (baroclinic, barotropic and Ekman types). While both the baroclinic (M_{yc}) and barotropic (M_{yD}) transports are geostrophic, the Ekman transport (M_{yDE}) is non geostrophic slope current due to the influence of wind stresses at the sea surface.

Though the mass continuity equation (40a) requires that the divergence of sum of the three types of transports in equation (60) is zero (or equal to rate at which water is

added at the surface), this need not hold good for each type of transport separately. So Fofonoff (1962) interprets the equation (52) following the approach of Sverdrup by cross differentiation and adding both the equations as :

$$f\left[\frac{\partial M_{xg}}{\partial x}+\frac{\partial M_{yg}}{\partial y}\right]+\beta M_{yg}=0 \;\ldots\ldots(61)$$

$$f\left[\frac{\partial M_{xE}}{\partial x}+\frac{\partial M_{yE}}{\partial y}\right]+\beta M_{yE}=\frac{\partial \tau_{My}}{\partial x}-\frac{\partial \tau_{Mx}}{\partial y}\;\ldots\ldots(62)$$

$$f\left[\frac{\partial M_{xD}}{\partial x}+\frac{\partial M_{yD}}{\partial y}\right]+\beta M_{yD}=-\rho' f\left[u_D\frac{\partial D}{\partial x}+v_D\frac{\partial D}{\partial y}\right]\ldots..(63)$$

Where $\rho'=\rho_D\left[1+\frac{p_D}{\alpha_D}\left(\frac{\partial \alpha}{\partial p}\right)_D\right]$

Combining all these equations of (61) to (63) we can write

$$\beta M_y=\nabla\times\tau_\eta-\nabla\times\tau_D-\rho' f\left[u_D\frac{\partial D}{\partial x}+v_D\frac{\partial D}{\partial y}\right]\ldots\ldots\ldots(64)$$

Here χ terms cancel each other while cross differentiating both the equations of (52). If the barotropic deep flow is assumed as zero and Tangential stresses near the bottom are also assumed as zero, equation (64) reduces to the Sverdrup's equation (38).

$$\therefore\ \beta M_y=\nabla_z\times\tau_0$$